A PROPERTY OF (v, k, λ)-DESIGNS

BY

O. MARRERO

ABSTRACT

It is shown that if the subsets $X_1, \ldots, X_v$ of a set X form a (v, k, λ)-design, then there does not exist another subset X_{v+1} of X having *any* cardinality k_1 and intersecting each of the X_j, $1 \leqq j \leqq v$, in *any* number λ_1 of elements, where $0 < k_1 < v$ and $0 < \lambda_1 < k$ (in order to avoid uninteresting cases).

Let $X = \{x_1, \cdots, x_v\}$ and let $X_1, \cdots, X_v$ be subsets of X. The subsets $X_1, \cdots, X_v$ are said to form a (v, k, λ)-*design* if

each X_j, $1 \leqq j \leqq v$, has k elements, any two distinct X_i, X_j, $1 \leqq i, j \leqq v$, intersect in λ elements, and $0 \leqq \lambda < k < v - 1$.

Such a design is completely determined by its *incidence matrix*; this is the $(0,1)$-matrix $A = [a_{ij}]$ defined by taking $a_{ij} = 1$ if $x_j \in X_i$ and $a_{ij} = 0$ if $x_j \notin X_i$. More information about these combinatorial designs is available, for example, in [1] and [2].

It is well known that if the subsets $X_1, \cdots, X_v$ of a set X form a (v, k, λ)-design, then it is not possible to find another subset X_{v+1} of X so that X_{v+1} has k elements and X_{v+1} intersects each X_j, $1 \leqq j \leqq v$, in λ elements. (This is a consequence, for example, of the well known "Fisher's inequality" for balanced incomplete block designs.) The purpose of this note is to prove the following more general result.

THEOREM. *Suppose the subsets $X_1, \cdots, X_v$ of a set X form a (v, k, λ)-design. Then there does not exist another subset X_{v+1} of X so that X_{v+1} has k_1 elements and X_{v+1} intersects each X_j, $1 \leqq j \leqq v$, in λ_1 elements, where $0 < k_1 < v$ and $0 < \lambda_1 < k$.*

Received March 6, 1972

It is known that no such subset X_{v+1}, as in the statement of the above theorem, can exist if one still allows X_{v+1} to have any cardinality $k_1, 0 < \lambda < k_1 < v$, but insists on requiring that X_{v+1} have λ elements in common with each X_j, $1 \leqq j \leqq v$; see, for example, Theorem 1.1 in [3].

The proof of the above theorem will be accomplished by showing that the corresponding incidence matrix cannot exist. Let A be the incidence matrix of a (v,k,λ)-design. For notation, $\boldsymbol{\sigma} \circ \boldsymbol{\eta}$ will denote the usual inner product of two vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$. It will be shown that it is not possible to find a v dimensional $(0,1)$-vector $\boldsymbol{\sigma}$ satisfying $0 < \boldsymbol{\sigma} \circ \boldsymbol{\sigma} = k_1 < v$ and $k > \boldsymbol{\sigma} \circ \boldsymbol{\eta}_r = \lambda_1 > 0$, for $r = 1, \cdots, v$, where $\boldsymbol{\eta}_r$ is the rth row vector of A.

Suppose it is possible to find such a vector $\boldsymbol{\sigma}$, and consider $\boldsymbol{\eta}_1, \cdots, \boldsymbol{\eta}_v$, $\boldsymbol{\sigma}$ as v dimensional vectors over the field of real numbers. There exist real numbers $a_1, \cdots, a_v$, not all 0, such that $\boldsymbol{\sigma} = \sum_{s=1}^{v} a_s \boldsymbol{\eta}_s$. Since each column of A has exactly k ones,

$$\boldsymbol{\sigma} = \sum_{s=1}^{v} a_s \boldsymbol{\eta}_s = (a_1^{(1)} + \cdots + a_k^{(1)}, \cdots, a_1^{(v)} + \cdots + a_k^{(v)}),$$

where $\{a_1^{(r)}, \cdots, a_k^{(r)}\}$ is a subsequence of $\{a_1, \cdots, a_v\}$ for $r = 1, \cdots, v$. Because $\boldsymbol{\sigma} \circ \boldsymbol{\eta}_r = \lambda_1$ for $1 \leqq r \leqq v$,

$$\sum_{r=1}^{v} \sum_{s=1}^{k} a_s^{(r)} = k_1 = \boldsymbol{\sigma} \circ \boldsymbol{\sigma} = \lambda_1 \sum_{s=1}^{v} a_s.$$

Thus, $k_1 \neq 0$ implies $\sum_{s=1}^{v} a_s \neq 0$. But, since each row of A has exactly k ones, it also must be true that

$$\sum_{r=1}^{v} \sum_{s=1}^{k} a_s^{(r)} = k \sum_{s=1}^{v} a_s.$$

Consequently, $k = \lambda_1$, which is contrary to the hypothesis.

REFERENCES

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FRANCIS MARION COLLEGE
FLORENCE, SOUTH CAROLINA